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A Multiverse Axiom Induction Framework

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Pluralism in action. Although the traditional platonist view of an immutable set reality waiting for discovery may still be implicitly dominant, in recent years a more pluralist perspective has gained ground. This development is not only a reaction to the wave of independence results obtained since the sixties, but also reflects the persisting difficulties to settle specific questions, like the continuum hypothesis, by adding more or less well motivated set-theoretic assumptions, e.g. large cardinal, determinacy, or forcing axioms. Pluralists assume that the topic of set theory is not some canonical, ontologically questionable "real" universe, but a multiverse, a multiplicity of possible universes, like suitable models of some basic set-theoretic axioms. The main goal is to investigate the properties and the diversity of its members, together with their relationships. Pluralism itself is pluralist - it may reflect ontological commitments, like the rejection of a realist philosophy, or epistemological modesty paying tribute to recognized limitations of meta-mathematical inquiry.

However, independently from philosophical attitudes, the multiverse paradigm offers a powerful methodological tool for investigating the conceptual foundations of set theory by guiding the search for and the evaluation of new axioms. This is the idea behind Friedman's pragmatic but systematic Hyperuniverse Program (HUP)[1], which tries to rely on objectively justifiable mechanisms for determining new axioms. In fact, a missing set heaven doesn't preclude a progressive, well-orchestrated addition of new reasonable set-theoretic commitments, reflecting the evolution of mathematical practice, knowledge, and needs. And even Platonism needs an effective exploration strategy to pass beyond the stage of vague or fortuitous insights.

Friedman's Hyperuniverse $\mathbb{H}$ consists of all the countable transitive models of ZFC, witnessing all the commonly discussed axioms (assuming consistency). This multiverse, taking advantage of the Loewenheim-Skolem theorem, may be considered unbiased in the context of first-order ZFC and is closed under all the common model constructions. The goal is to identify preferred universes whose shared consequences may then point towards new reasonable axiom candidates. More concretely, one considers general, intuitively appealing set-theoretical principles, like Maximality, and uses them to induce formal criteria over $\mathbb{H}$, like variants and combinations of horizontal/power set and vertical maximality. The resulting constraints, insofar compatible with set-theoretic practice (e.g. the consistency of relevant large cardinal concepts), can then be exploited to specify a collection of preferred universes $\mathbb{H}_{pr}$ whose theory is meant to justify or filter out axioms.

[1] S. Friedman, T. Arragoni. The Hyperuniverse Program. *Bulletin of Symbolic Logic*: 19(1):77-96, 2013.

A formal framework for axiom induction The goal of the present work is to develop an abstract inferential framework inspired by the HUP whose inference method instantiations may be used for identifying new set-theoretic axioms, and in addition, can themselves be compared and evaluated. The idea is to investigate natural parametrized "inductive" inference relations $\mid\!\sim^{\in}$ which associate with any suitable axiom system Σ, usually $ZFC + X$ for recursive X, reasonable candidates for new set-theoretic axioms or truths. To specify such a $\mid\!\sim^{\in}$ we can use as parameters formal specifications of the multiverse, of set-theoretic beliefs/demands, and of philosophical/methodological desiderata. Rationality postulates for the inference relations and their parameters may then offer additional criteria for judging axiom generation, transcending the HUP.

Axiom induction is meant to extend a given axiom system Σ, known to provide only a partial description of some topic of interest, by plausible new axiom candidates. The completion of axiomatic set theories by large cardinal axioms may be the most prominent example. Let $L = L(\in)$ be the first-order language for set-theory and $\vdash$ be classical first-order inference. The central concept is that of an inference relation $\mid\!\sim^{\in}_{\Delta} \subseteq 2^L \times L$ linking an initial collection of accepted set-theoretic axioms Σ to presumably reasonable set-theoretic constraints or axiom candidates. The parameter Δ encodes additional information about the multiverse, mathematical practice, or philosophical/technical/practical desiderata. The general scheme is thus $\Sigma \mid\!\sim^{\in}_{\Delta} \psi$.

Let us take a look at possible basic ingredients on the input/output side. In practice, a multiverse like $\mathbb{H}$ has to be anchored in a set-theoretic environment $\mathcal{S} = (S, \in)$. While a genuinely pluralist perspective may prefer weak conditions, in the context of axiom induction for ZFC, the reference to set-theoretic practice requires strong consistency assumptions so that minimalism seems impractical and misguided. Let $\Gamma \subseteq L$ be the collection of axioms - or more generally, a collection of axiom sets - transcending ZFC (large cardinals, forcing, determinacy, constructibility, ...) considered relevant, and thus to be instantiated in the multiverse. For each $\gamma \in \Gamma$, let γ^+ express the existence of a transitive model of $ZF + \gamma$ and let $\Gamma^+ = \{\gamma^+ \mid \gamma \in \Gamma\}$. Note that we do not ask Γ to be consistent with ZFC. There may be incompatible but a priori individually justifiable axiomatic choices, at least realizable in inner models (like the inner model hypothesis and "$\exists$ inaccessible cardinals", or ZFC and AD). Hence, if $\hat{T} \subseteq L$ denotes the conditions imposed upon $\mathcal{S}$, to ensure the existence of a rich and axiom-induction-friendly multiverse, we stipulate the consistency of $ZFC \cup \Gamma^+ \subseteq \hat{T}$. $\mathcal{S} \models \hat{T}$ figures as an implicit background universe hosting the transitive sets representing the universes meant to populate the multiverse.

In the context of $\mathcal{S} \models \hat{T}$ and a reference axiom system $\Sigma = ZFC + X$, a multiverse $\mathbb{MV}$ is a set/class of transitive set models of Σ, called universes, defined by a formula (set) $\Phi_{\mathbb{MV}}$. We impose Σ-*Representativity*, i.e. every theory realized by a model of Σ in $\mathcal{S}$ has to be realized in $\mathbb{MV}$ as well, and Σ-*closure*, i.e. for each universe in $\mathbb{MV}$, all its inner Σ-models are also in $\mathbb{MV}$. Because we seek a general framework able to handle also axioms formulated in infinitary or higher-order languages, we cannot just focus on countable transitive models, as Friedman's Hyperuniverse does. To honour the universality and unity of set theory, we also require *Directedness*: for all universes M_1, M_2, there is an universe M with $M_1 \cup M_2 \subseteq M$. The hyperuniverse $\mathbb{H}$, which we would tend to consider the minimal reasonable multiverse, and the maximal multiverse $Mod(\Sigma)$ obviously verify these three conditions. Further desiderata may be imposed.

A more pragmatic demand is to require new candidate axioms to be consistent with any established maximality-oriented axiom (forcing, large cardinals, not constructibility). We can describe these by a - possibly ZFC-inconsistent - subset $\Gamma^* \subseteq \Gamma$ of the set of relevant axioms.

This gives us four syntactic parameters for axiom induction: the background theory $\hat{T}$, the multiverse specification $\Phi_{\mathbb{MV}}$, the relevant axioms Γ, and the established axioms Γ^*. If we maximize the multiverse ($\Phi_{\mathbb{MV}} = \Sigma$) and presuppose that the inner-model-consistency assumptions are in Σ, i.e. $\Gamma^+ \subseteq \Sigma$, we are left with $\hat{T}$ and Γ^*. Because the part of $\hat{T}$ depending on Σ (asking for an inner set model of Σ) is given by Σ, we may replace $\hat{T} = \hat{T}_\Sigma$ by its non-dependent part $\hat{T}^-$. Without an obvious reason why the theory of the background universe should be stronger than necessary relative to Σ, we could even assume that $\hat{T}^- = \emptyset$.

The most natural way to specify "preferred universes" is to choose them according to some partial pre-order on universes based on the theories or the structures of the models in the multiverse. The natural large cardinal hierarchies, the amount of horizontal or vertical reflection, the inner-model ordering, and others, which can again be aggregated in different ways, they all provide pre-orders implementing conceptual desiderata. So we may assume that axiom induction can also be informed by one or several pre-orders $\preceq$ over universes. A simple parameter context would be $\Delta = (\Gamma^*, \preceq)$.

Reasonable inference notions. What do we know about the $\mid\!\sim^{\in}_{\Delta}$ and which principles should they satisfy? A characteristic of inductive reasoning is its nonmonotonicity: the addition of new premises may require the retraction of previous conclusions ψ, thereby invalidating the monotonicity of classical logic. A new axiom φ added for extrinsic reasons to Σ may well reject ψ. Since the early 80s, such defeasible consequence relations have been heavily investigated in AI and applied logic[2], where they are used to model and analyze commonsense reasoning. If axiom induction is just driven by preorders over the multiverse, the resulting $\mid\!\sim^{\in}_{\Delta}$ are well-behaved preferential consequence relations which verify for instance *Cumulativity*, a weakening of monotonicity and transitivity: If $\Sigma \mid\!\sim \psi$, then $\Sigma \mid\!\sim \varphi$ iff $\Sigma \cup \{\psi\} \mid\!\sim \varphi$.

If this fails, it is possible that two axioms are suggested and can even be consistently added, while adding just one may effectively block the other one. It is at least questionable whether such an inference notion is able to guide us to any coherent new axiomatic picture (even admitting the possibility of bifurcation). Cumulativity may not be a necessary condition for axiom induction, but it is certainly a desirable one. The situation can easily deteriorate. If we enforce the consistency conditions resulting from some Γ^*, then basic principles like *Right conjunction*, i.e. $\Sigma \mid\!\sim \varphi, \psi$ implies $\Sigma \mid\!\sim \varphi \wedge \psi$, are violated as well. By itself, this lack of global coherence is not necessarily tragic because it might be enough if axiom induction produces good individual candidates. The problem is however that we are dealing with an incremental dynamic process, where incoherence may stay hidden and the order of addition may affect the results. An analysis of the abstract properties of these inference mechanisms is therefore recommendable to allow a better choice of methods and a better understanding of the risks of misguided axiom addition.

[2] D. Gabbay et al. (eds.). Handbook of Logic in Artificial Intelligence and Logic Programming, Vol. 3: Nonmonotonic Reasoning and Uncertain Reasoning, Oxford University Press, Oxford, 1994.